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Semistable and numerically effective principal (Higgs) bundles [☆]

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Abstract

We study Miyaoka-type semistability criteria for principal Higgs G -bundles \mathcal{E} on complex projective manifolds of any dimension. We prove that \mathcal{E} has the property of being semistable after pullback to any projective curve if and only if certain line bundles, obtained from some characters of the parabolic subgroups of G , are numerically effective. One also proves that these conditions are met for semistable principal Higgs bundles whose adjoint bundle has vanishing second Chern class.

In a second part of the paper, we introduce notions of numerical effectiveness and numerical flatness for principal (Higgs) bundles, discussing their main properties. For (non-Higgs) principal bundles, we show that a numerically flat principal bundle admits a reduction to a Levi factor which has a flat Hermitian–Yang–Mills connection, and, as a consequence, that the cohomology ring of a numerically flat principal bundle with coefficients in \mathbb{R} is trivial. To our knowledge this notion of numerical effectiveness is new even in the case of (non-Higgs) principal bundles.

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1. Introduction

In 1987 Miyaoka gave a criterion for the semistability of a vector bundle V on a projective curve in terms of the numerical effectiveness of a suitable divisorial class (the relative anticanonical divisor of the projectivisation $\mathbb{P}V$ of V) [18]. Recently several generalisations of this criterion have been formulated [10,4,6], dealing with principal bundles, higher-dimensional varieties, and considering also the case of bundles on compact Kähler manifolds.

In this paper we prove a Miyaoka-type criterion for principal Higgs bundles on complex projective manifolds. Let us give a rough anticipation of this result. Given a principal Higgs G -bundle $\mathfrak{E} = (E, \phi)$ on a complex projective manifold X , with Higgs field ϕ , and a parabolic subgroup P of G , we introduce a subscheme $\mathfrak{R}_P(E, \phi)$ of the total space of the bundle $E/P \rightarrow X$ whose sections parametrise reductions of the structure group G to P that are compatible with the Higgs field ϕ . Then in Theorem 4.7 we prove the equivalence of the following conditions: for every reduction of G to a parabolic subgroup P which is compatible with the Higgs field, and every dominant character of P , a certain associated line bundle on $\mathfrak{R}_P(E, \phi)$ is numerically effective; the pullback $f^*\mathfrak{E}$ is semistable for any morphism $f: C \rightarrow X$, where C is any smooth projective curve. One also shows that both conditions are met when \mathfrak{E} is a semistable principal Higgs bundle such that $c_2(\text{Ad}(E)) = 0$.

In a second part of this paper, we define notions of numerical effectiveness and numerical flatness which are appropriate for principal Higgs bundles. It is known [12] that a numerically flat vector bundle admits a filtration whose quotients are stable Hermitian flat vector bundles. In Section 6 we prove that to a numerically flat principal (non-Higgs) bundle one can associate a principal bundle, whose structure group is the Levi factor of a parabolic subgroup of G , which is polystable, and admits a flat “Hermitian” connection. This implies that the characteristic ring (with coefficients in \mathbb{R}) of the principal bundle vanishes.

Section 7 develops some Tannakian considerations; basically we show the equivalence of proving our Theorem 4.7 for principal Higgs bundles or for Higgs vector bundles.

In Appendix A we offer a resume of our previous work on Higgs vector bundles [8–10], on which some parts of the present paper rely quite heavily.

As a principal Higgs bundle with zero Higgs field is exactly a principal bundle, all results we prove in this paper hold true for principal bundles. In this way we mostly recover well-known results or some of the results in [4,5] with their proofs, at other times we provide simpler de-

mostrations, while at times the results are altogether new. The notion of numerical effectiveness we introduce is, on the other hand, new also for the case of principal bundles.

2. Semistable principal bundles

In this short section we recall some basics about principal bundles, notably the definition of (semi)stable principal bundle (basic references about this topic are [20,3]). Let X be a smooth complex projective variety, G a complex reductive algebraic group, and $\pi: E \rightarrow X$ a principal G -bundle on X . If $\rho: G \rightarrow \text{Aut}(Y)$ is a representation of G as automorphisms of a variety Y , we may construct the associated bundle $E(\rho) = E \times_{\rho} Y$, the quotient of $E \times Y$ under the action of G given by $(u, y) \mapsto (ug, \rho(g^{-1})y)$ for $g \in G$. If $Y = \mathfrak{g}$ is the Lie algebra of G , and ρ is the adjoint action of G on \mathfrak{g} , one gets the adjoint bundle of E , denoted by $\text{Ad}(E)$. Another important example is obtained when ρ is given by a group homomorphism $\lambda: G \rightarrow G'$; in this case the associated bundle $E' = E \times_{\lambda} G'$ is a principal G' -bundle. We say that the structure group G of E has been extended to G' .

If E is a principal G -bundle on X , and F a principal G' -bundle on X , a morphism $E \rightarrow F$ is a pair (f, f') , where $f': G \rightarrow G'$ is a group homomorphism, and $f: E \rightarrow F$ is a morphism of bundles on X which is f' -equivariant, i.e., $f(ug) = f(u)f'(g)$. Note that this induces a vector bundle morphism $\tilde{f}: \text{Ad}(E) \rightarrow \text{Ad}(F)$ given by $\tilde{f}(u, \alpha) = (f(u), f'_*(\alpha))$, where $f'_*: \mathfrak{g} \rightarrow \mathfrak{g}'$ is the morphism induced on the Lie algebras. As an example, consider a principal G -bundle E , a group homomorphism $\lambda: G \rightarrow G'$, and the extended bundle E' . There is a natural morphism $(f, \lambda): E \rightarrow E'$, where $f = \text{id} \times \lambda$ if we identify E with $E \times_G G$.

If K is a closed subgroup of G , a *reduction* of the structure group G of E to K is a principal K -bundle F over X together with an injective K -equivariant bundle morphism $F \rightarrow E$. Let $E(G/K)$ denote the bundle over X with standard fibre G/K associated to E via the natural action of G on the homogeneous space G/K . There is an isomorphism $E(G/K) \simeq E/K$ of bundles over X . Moreover, the reductions of the structure group of E to K are in a one-to-one correspondence with sections $\sigma: X \rightarrow E(G/K) \simeq E/K$.

We first recall the definition of semistable principal bundle when the base variety X is a curve. Let $T_{E/K, X}$ be the vertical tangent bundle to the bundle $\pi_K: E/K \rightarrow X$.

Definition 2.1. Let E be a principal G -bundle on a smooth connected projective curve X . We say that E is stable (semistable) if for every proper parabolic subgroup $P \subset G$, and every reduction $\sigma: X \rightarrow E/P$, the pullback $\sigma^*(T_{E/P, X})$ has positive (nonnegative) degree.

When X is a higher-dimensional variety, the definition must be somewhat refined; the introduction of an open dense subset whose complement has codimension at least two should be compared with the definition of (semi)stable vector bundle, which involves non-locally free subsheaves (which are subbundles exactly on open subsets of that kind).

Definition 2.2. Let X be a polarised smooth projective variety. A principal G -bundle E on X is stable (semistable) if and only if for any proper parabolic subgroup $P \subset G$, any open dense subset $U \subset X$ such that $\text{codim}(X - U) \geq 2$, and any reduction $\sigma: U \rightarrow (E/P)|_U$ of G to P on U , one has $\deg \sigma^*(T_{E/P, X}) > 0$ ($\deg \sigma^*(T_{E/P, X}) \geq 0$).

Here it is important that the smoothness of X guarantees that a line bundle defined on an open dense subset of X , whose complement has codimension 2 at least, extends uniquely to the whole

of X , so that we may consistently consider its degree. This is discussed in detail in [21], see also [17, Chapter V].

3. Principal Higgs bundles

We switch now to principal Higgs bundles. Let X be a smooth complex projective variety, and G a reductive complex algebraic group. If E is a principal G -bundle on X , $\text{Ad}(E)$ is its adjoint bundle, and ϕ, ψ are global sections of $\text{Ad}(E) \otimes \Omega_X^1$, we can define a section $[\phi, \psi]$ of $\text{Ad}(E) \otimes \Omega_X^2$ by combining the bracket $[\cdot, \cdot]: \text{Ad}(E) \otimes \text{Ad}(E) \rightarrow \text{Ad}(E)$ with the natural morphism $\Omega_X^1 \otimes \Omega_X^1 \rightarrow \Omega_X^2$.

Definition 3.1. A principal Higgs G -bundle \mathfrak{E} is a pair (E, ϕ) , where E is a principal G -bundle, and ϕ is a global section of $\text{Ad}(E) \otimes \Omega_X^1$ such that $[\phi, \phi] = 0$.

When G is the general linear group, under the identification $\text{Ad}(E) \simeq \text{End}(V)$, where V is the vector bundle corresponding to E , this agrees with the usual definition of Higgs vector bundle.

Definition 3.2. A principal Higgs G -bundle $\mathfrak{E} = (E, \phi)$ is *trivial* if E is trivial, and $\phi = 0$.

A morphism between two principal Higgs bundles $\mathfrak{E} = (E, \phi)$ and $\mathfrak{E}' = (E', \phi')$ is a principal bundle morphism $f: E \rightarrow E'$ such that $(f_* \times \text{id})(\phi) = \phi'$, where $f_*: \text{Ad}(E) \rightarrow \text{Ad}(E')$ is the induced morphism between the adjoint bundles.

We introduce the notion of extension of the structure group for a principal Higgs G -bundle $\mathfrak{E} = (E, \phi)$. Given a group homomorphism $\lambda: G \rightarrow G'$, we consider the extended principal bundle E' . The group G acts on the Lie algebra \mathfrak{g}' of G' via the homomorphism λ (and the adjoint action of G'), and the \mathfrak{g}' -bundle associated to E via the adjoint action of G' is isomorphic to $\text{Ad}(E')$. In this way the Higgs field of \mathfrak{E} induces a Higgs field for \mathfrak{E}' . More generally, if $\rho: G \rightarrow \text{Aut}(V)$ is a linear representation of G , the Higgs field of \mathfrak{E} induces a Higgs field for the associated vector bundle $E \times_\rho V$.

If \mathfrak{E} is a principal Higgs G -bundle, we denote by $\text{Ad}(\mathfrak{E})$ the Higgs vector bundle given by the adjoint bundle $\text{Ad}(E)$ equipped with the induced Higgs morphism.

Let K be a closed subgroup of G , and $\sigma: X \rightarrow E(G/K) \simeq E/K$ a reduction of the structure group of E to K . So one has a principal K -bundle F_σ on X and a principal bundle morphism $i_\sigma: F_\sigma \rightarrow E$ inducing an injective morphism of bundles $\text{Ad}(F_\sigma) \rightarrow \text{Ad}(E)$. Let $\Pi_\sigma: \text{Ad}(E) \otimes \Omega_X^1 \rightarrow (\text{Ad}(E)/\text{Ad}(F_\sigma)) \otimes \Omega_X^1$ be the induced projection.

Definition 3.3. A section $\sigma: X \rightarrow E/K$ is a *Higgs reduction* of (E, ϕ) if $\phi \in \ker \Pi_\sigma$.

When this happens, the reduced bundle F_σ is equipped with a Higgs field ϕ_σ compatible with ϕ (i.e., $(F_\sigma, \phi_\sigma) \rightarrow (E, \phi)$ is a morphism of principal Higgs bundles).

Remark 3.4. Let us again consider the case when G is the general linear group $GL(n, \mathbb{C})$, and let us assume that K is a maximal parabolic subgroup, so that G/K is the Grassmann variety $\text{Gr}_k(\mathbb{C}^n)$ of k -dimensional quotients of \mathbb{C}^n for some k . If V is the vector bundle corresponding to E , a reduction σ of G to K corresponds to a rank $n - k$ subbundle W of V , and the fact that σ is a Higgs reduction means that W is ϕ -invariant.

The choice of ϕ singles out a subscheme of the variety E/K , which describes the Higgs reductions of the pair (E, ϕ) . Let E_K denote the principal K -bundle $E \rightarrow E/K$. Since the vertical tangent bundle $T_{E/K, X}$ is the bundle associated to E_K via the adjoint action of K on the quotient $\mathfrak{g}/\mathfrak{k}$, and $\pi_K^* \text{Ad}(E)$ is the bundle associated to E_K via the adjoint action of K on \mathfrak{g} , there is a natural morphism $\eta: \pi_K^* \text{Ad}(E) \rightarrow T_{E/K, X}$. Then ϕ determines a section $\eta(\phi) := (\eta \otimes \text{id})(\pi_K^* \phi)$ of $T_{E/K, X} \otimes \Omega_{E/K}^1$.

Definition 3.5. The *scheme of Higgs reductions* of $\mathfrak{E} = (E, \phi)$ to K is the closed subscheme $\mathfrak{R}_K(\mathfrak{E})$ of E/K given by the zero locus of $\eta(\phi)$.

Remark 3.6. The Higgs field of \mathfrak{E} induces a Higgs field on the restriction of E_K to $\mathfrak{R}_K(\mathfrak{E})$; we denote by \mathfrak{E}_K the resulting principal Higgs K -bundle.

The construction of the scheme of Higgs reductions is compatible with base change. Let us recall that given a principal Higgs G -bundle $\mathfrak{E} = (E, \phi)$ over X , and a morphism $f: Y \rightarrow X$, the pullback Higgs bundle $f^* \mathfrak{E}$ is the pullback principal bundle $f^* E$ equipped with a Higgs field obtained by combining the pullback morphism

$$\text{Ad}(f^* E) \simeq f^* \text{Ad}(E) \rightarrow \text{Ad}(f^* E) \otimes f^* \Omega_X^1$$

with the natural morphism $f^* \Omega_X^1 \rightarrow \Omega_Y^1$. The above mentioned compatibility means that, if f is a morphism of smooth complex projective varieties, then $\mathfrak{R}_K(f^*(\mathfrak{E})) \simeq Y \times_X \mathfrak{R}_K(\mathfrak{E})$. By construction, $\sigma: X \rightarrow E(G/K) \simeq E/K$ is a Higgs reduction if and only if it takes values in the subscheme $\mathfrak{R}_K(\mathfrak{E}) \subset E/K$. Moreover the scheme of Higgs reductions is compatible with morphisms of principal Higgs bundles. This means that if $\mathfrak{E} = (E, \phi)$ is a principal Higgs G -bundle, $\mathfrak{E}' = (E', \phi')$ a principal Higgs G' -bundle, $\psi: G \rightarrow G'$ is a group homomorphism, and $f: \mathfrak{E} \rightarrow \mathfrak{E}'$ is a ψ -equivariant morphism of principal Higgs bundles, then for every closed subgroup $K \subset G$ the induced morphism $E/K \rightarrow E'/K'$, where $K' = \psi(K)$, maps $\mathfrak{R}_K(\mathfrak{E})$ into $\mathfrak{R}_{K'}(\mathfrak{E}')$.

Also, one should note that the scheme of Higgs reductions is in general singular, so that in order to consider Higgs bundles on it one needs to use the theory of the de Rham complex for arbitrary schemes, as developed by Grothendieck [15].

For the time being we restrict our attention to the case when X is a curve. We start by introducing a notion of semistability for principal Higgs bundles (which is equivalent to the one given in Definition 4.6 in [2]).

Definition 3.7. Let X be a smooth projective curve. A principal Higgs G -bundle $\mathfrak{E} = (E, \phi)$ is stable (resp. semistable) if for every parabolic subgroup $P \subset G$ and every Higgs reduction $\sigma: X \rightarrow \mathfrak{R}_P(\mathfrak{E})$ one has $\deg \sigma^*(T_{E/P, X}) > 0$ (resp. $\deg \sigma^*(T_{E/P, X}) \geq 0$).

Lemma 3.8. Let $f: X' \rightarrow X$ be a nonconstant morphism of smooth projective curves, and \mathfrak{E} a principal Higgs G -bundle on X . The pullback Higgs bundle $f^* \mathfrak{E}$ is semistable if and only if \mathfrak{E} is.

Proof. As we shall prove in Lemma 4.3 in the case of X of arbitrary dimension, a principal Higgs bundle \mathfrak{E} is semistable if and only if the adjoint Higgs bundle $\text{Ad}(\mathfrak{E})$ is semistable (as a

Higgs vector bundle). In view of this result, our claim reduces to the analogous statement for Higgs vector bundles, which was proved in [10]. \square

If $\mathfrak{E} = (E, \phi)$ is a principal Higgs G -bundle on X , and K is a closed subgroup of G , we may associate with every character χ of K a line bundle $L_\chi = E \times_\chi \mathbb{C}$ on E/K , where we regard E as a principal K -bundle on E/K . An elegant way to state results about reductions is to introduce the notion of *slope* of a reduction: we call μ_σ , the slope of a Higgs reduction σ , the group homomorphism $\mu_\sigma : \mathcal{X}(K) \rightarrow \mathbb{Q}$ (where $\mathcal{X}(K)$ is the group of characters of K) which to any character χ associates the degree of the line bundle $\sigma^*(L_\chi^*)$.

By a simple modification of the proof of Lemma 2.1 of [20] we can extend it to Higgs bundles. If \mathfrak{g} is the Lie algebra of G and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is its semisimple part, let $\alpha_1, \dots, \alpha_r$ be simple roots of \mathfrak{g}' , and let $\lambda_1, \dots, \lambda_r$ be the corresponding system of fundamental weights of \mathfrak{g}' . Given a parabolic subgroup $P \subset G$, a character $\chi : P \rightarrow \mathbb{C}^*$ is said to be *dominant* if it is a linear combination of the fundamental weights λ_i with nonnegative coefficients. Such a character is trivial on the centre $Z(G)$ of G .

Lemma 3.9. *A principal Higgs G -bundle $\mathfrak{E} = (E, \phi)$ is semistable if and only if for every parabolic subgroup $P \subset G$, every nontrivial dominant character χ of P , and every Higgs reduction $\sigma : X \rightarrow \mathfrak{R}_P(\mathfrak{E})$, one has $\mu_\sigma(\chi) \geq 0$.*

Proof. We may at first assume that P is a maximal parabolic subgroup corresponding to a root α_i . It has been proven in [20, Lemma 2.1] that the determinant of the vertical tangent bundle $T_{E/P, X}$ is associated to the principal P -bundle $E \rightarrow E/P$ via a character that may be expressed as $\mu = -m\lambda_i$, where λ_i is the weight corresponding to α_i , and $m \geq 0$. Thus, if $\sigma : X \rightarrow \mathfrak{R}_P(\mathfrak{E})$ is a Higgs reduction, $\deg(\sigma^*(L_\mu^*)) \geq 0$ if and only if $\deg \sigma^*(T_{E/P, X}) \geq 0$.

If P is not maximal, any dominant character of P is a sum of dominant characters χ_k of the maximal parabolic subgroups P_k that contain P , with $k = 1, \dots, m$ for some m . Moreover, any Higgs reduction $\sigma : X \rightarrow \mathfrak{R}_P(\mathfrak{E})$ induces a Higgs reduction $\sigma_k : X \rightarrow \mathfrak{R}_{P_k}(\mathfrak{E})$. If \mathfrak{E} is semistable, we have $\deg \sigma_k^*(L_{\chi_k})^* \geq 0$. Since $\sigma^*(L_\chi) \simeq \sigma_1^*(L_{\chi_1}) \otimes \dots \otimes \sigma_m^*(L_{\chi_m})$, we have $\mu_\sigma(\chi) \geq 0$. \square

We may now state and prove a Miyaoka-type semistability criterion for principal Higgs bundles over projective curves. This generalises Proposition 2.1 of [4], and, of course, Miyaoka's original criterion in [18].

Theorem 3.10. *A principal Higgs G -bundle $\mathfrak{E} = (E, \phi)$ on a smooth projective curve X is semistable if and only if for every parabolic subgroup $P \subset G$, and every nontrivial dominant character χ of P , the line bundle L_χ^* restricted to $\mathfrak{R}_P(\mathfrak{E})$ is nef.*

Proof. Assume that \mathfrak{E} is semistable and that $L_{\chi|_{\mathfrak{R}_P(\mathfrak{E})}}^*$ is not nef. Then there is an irreducible curve $Y \subset \mathfrak{R}_P(\mathfrak{E})$ such that $[Y] \cdot c_1(L_\chi^*) < 0$. Since χ is dominant, the line bundle L_χ^* is nef when restricted to a fibre of the projection $E/P \rightarrow X$, so that the curve Y cannot be contained in such a fibre. Then Y surjects onto X . One can choose a morphism of smooth projective curves $h : X' \rightarrow X$ such that $\tilde{Y} = X' \times_X Y$ is a curve in $h^*(\mathfrak{R}_P(\mathfrak{E}))$, whose irreducible components are smooth and map isomorphically to X' (i.e., $\tilde{Y} \rightarrow X'$ is a split unramified covering). By Lemma 3.8, the pullback of \mathfrak{E} to \tilde{Y} is semistable. We may think of the irreducible components of \tilde{Y} as images of sections σ_j of $h^*(\mathfrak{R}_P(\mathfrak{E}))$. By Lemma 3.9 this implies that $\deg \sigma_j^*(L')^* \geq 0$,

where L' is the pullback of L_χ to $h^*(\mathfrak{R}_P(\mathfrak{E}))$. This in turn implies $[Y] \cdot c_1(L_\chi^*) \geq 0$, but this contradicts our assumption.

The converse is obvious in view of Lemma 3.9. \square

Remark 3.11. Let G be the linear group $Gl(n, \mathbb{C})$. If $\mathfrak{E} = (E, \phi)$ is a principal Higgs G -bundle, and V is the rank n vector bundle corresponding to E , then the identification $\text{Ad}(E) \simeq \text{End}(V)$ makes ϕ into a Higgs morphism $\tilde{\phi}$ for V . The semistability of \mathfrak{E} is equivalent to the semistability of the Higgs vector bundle $(V, \tilde{\phi})$.

If P_k is a maximal parabolic subgroup of $Gl(n, \mathbb{C})$, E/P_k is the Grassmann bundle $\text{Gr}_k(V)$ of rank k locally free quotients of V . Then Theorem 3.10 corresponds to the result given in [10], according to which (V, ϕ) is semistable if and only if certain numerical classes θ_k in a closed subscheme of $\text{Gr}_k(V)$ are nef (see Appendix A and [10,8,9] for details).

4. The higher-dimensional case

In this section we consider the case of a base variety X which is a complex projective manifold of any dimension. Let X be equipped with a polarisation H , and let G be a reductive complex algebraic group.

Definition 4.1. A principal Higgs G -bundle $\mathfrak{E} = (E, \phi)$ is stable (resp. semistable) if and only if for any proper parabolic subgroup $P \subset G$, any open dense subset $U \subset X$ such that $\text{codim}(X - U) \geq 2$, and any Higgs reduction $\sigma: U \rightarrow \mathfrak{R}_P(\mathfrak{E})|_U$ of G to P on U , one has $\deg \sigma^*(T_{E/P, X}) > 0$ (resp. $\deg \sigma^*(T_{E/P, X}) \geq 0$).

Remark 4.2. The arguments in the proof of Lemma 3.9 go through also in the higher-dimensional case, allowing one to show that a principal Higgs G -bundle \mathfrak{E} is semistable (stable)—according to Definition 4.1—if and only if for any proper parabolic subgroup $P \subset G$, any nontrivial dominant character χ of P , any open dense subset $U \subset X$ such that $\text{codim}(X - U) \geq 2$, and any Higgs reduction $\sigma: U \rightarrow \mathfrak{R}_P(\mathfrak{E})|_U$ of G to P on U , the line bundle $\sigma^*(L_\chi^*)$ has nonnegative (positive) degree.

It is known that certain extensions of the structure group of a semistable principal bundle are still semistable [19], and that a principal bundle is semistable if and only if its adjoint bundle is [20]. The same is true in the Higgs case.

Lemma 4.3.

- (i) A principal Higgs bundle \mathfrak{E} is semistable if and only if $\text{Ad}(\mathfrak{E})$ is semistable (as a Higgs vector bundle).
- (ii) A principal Higgs G -bundle $\mathfrak{E} = (E, \phi)$ is semistable if and only if for every linear representation $\rho: G \rightarrow \text{Aut}(V)$ of G such that $\rho(Z(G)_0)$ is contained in the centre of $\text{Aut}(V)$, the associated Higgs vector bundle $\mathfrak{V} = \mathfrak{E} \times_\rho V$ is semistable (here $Z(G)_0$ is the component of the centre of G containing the identity).

Remark 4.4. If G is the general linear group $Gl(n, \mathbb{C})$, the first claim holds true quite trivially: \mathfrak{E} is semistable if and only if the corresponding Higgs vector bundle \mathfrak{V} is semistable, and one knows that $\text{Ad}(\mathfrak{E}) \simeq \text{End}(\mathfrak{V})$ is semistable if and only if \mathfrak{V} is.

Proof. The first claim is Lemma 4.7 of [2]. The second claim is proved as in Lemma 1.3 of [1]. \square

Proposition 4.5. *Let $\lambda: G \rightarrow G'$ be a homomorphism of connected reductive algebraic groups which maps the connected component of the centre of G into the connected component of the centre of G' . If \mathfrak{E} is a semistable principal Higgs G -bundle, and \mathfrak{E}' is obtained by extending the structure group G to G' by λ , then \mathfrak{E}' is semistable.*

Proof. By composing the adjoint representation of G' with the homomorphism λ we obtain a representation $\rho: G \rightarrow \text{Aut}(\mathfrak{g}')$; the principal Higgs bundle obtained by extending the structure group of \mathfrak{E} to $\text{Aut}(\mathfrak{g}')$ is the bundle of linear frames of $\text{Ad}(\mathfrak{E}')$ with its natural Higgs field. By Lemma 4.3, this bundle is semistable, so that $\text{Ad}(\mathfrak{E}')$ is semistable as well. Again by Lemma 4.3, \mathfrak{E}' is semistable. \square

Remark 4.6. A notion of semistability for principal Higgs bundles was introduced by Simpson in [22]. Let us say that a principal Higgs G -bundle \mathfrak{E} is *Simpson-semistable* if there exists a faithful linear representation $\rho: G \rightarrow \text{Aut}(W)$ such that the associated Higgs vector bundle $\mathfrak{W} = \mathfrak{E} \times_{\rho} W$ is semistable. It is not difficult to show that Simpson-semistability implies semistability; indeed if \mathfrak{E} is Simpson-semistable, and ρ is a faithful linear representation such that \mathfrak{W} is semistable, then $\text{End}(\mathfrak{W})$, with its natural Higgs bundle structure, is semistable. But $\text{End}(\mathfrak{W}) \simeq \text{Ad}(GL(\mathfrak{W}))$, and $\text{Ad}(E)$ is a subbundle of $\text{Ad}(GL(\mathfrak{W}))$. Since both $\text{Ad}(E)$ and $\text{Ad}(GL(\mathfrak{W}))$ have vanishing first Chern class, $\text{Ad}(E)$ is semistable, so that E is semistable as well.

The contrary is not true, even in the case of ordinary (non-Higgs) principal bundles (in which case of course our definition coincides with Ramanan's classical definition of stability for principal bundles [20]). Indeed, if T is a torus in $GL(n, \mathbb{C})$, any principal T -bundle E is stable. However the vector bundle associated to it by the natural inclusion $T \hookrightarrow GL(n, \mathbb{C})$ (a direct sum of line bundles) may fail to be semistable, in which case E cannot be Simpson-semistable. (Note indeed that this inclusion, regarded as a linear representation of T , does not satisfy the condition in part (ii) of Lemma 4.3 unless $n = 1$.) A point in favour of the definition we choose is that it is compatible with the Hitchin–Kobayashi correspondence for principal bundles, which states that a principal G -bundle E , where G is a connected reductive complex group, is polystable if and only if it admits a reduction of the structure group to the maximal compact subgroup K of G such that the mean curvature of the unique connection on E compatible with the reduction takes values in the centre of the Lie algebra of K [21]. (We shall recall the definition of polystability of a principal Higgs bundle in Section 6.)

We can now prove a version of Miyaoka's semistability criterion which works for principal Higgs bundles on projective varieties of any dimension.

Theorem 4.7. *Let \mathfrak{E} be a principal Higgs G -bundle $\mathfrak{E} = (E, \phi)$ on X . Consider the following conditions:*

- (i) *for every parabolic subgroup $P \subset G$ and any nontrivial dominant character χ of P , the line bundle L_{χ}^* restricted to $\mathfrak{R}_P(\mathfrak{E})$ is numerically effective;*

- (ii) for every morphism $f : C \rightarrow X$, where C is a smooth projective curve, the pullback $f^*(\mathfrak{E})$ is semistable;
- (iii) \mathfrak{E} is semistable and $c_2(\mathrm{Ad}(E)) = 0$ in $H^4(X, \mathbb{R})$.

Then conditions (i) and (ii) are equivalent, and they are both implied by condition (iii).

Proof. Assume that condition (i) holds, and let $f : C \rightarrow X$ be as in the statement. The line bundle L'_χ on $f^*(E)/P$ given by the character χ is a pullback of L_χ . Then $L'_{\chi|_{\mathfrak{R}_P(f^*\mathfrak{E})}}$ is nef, so that by Theorem 3.10, $f^*(\mathfrak{E})$ is semistable. Thus (i) implies (ii).

We show now that (ii) implies (i). Let C' be a curve in $\mathfrak{R}_P(\mathfrak{E})$. If it is contained in a fibre of the projection $\pi_P : \mathfrak{R}_P(\mathfrak{E}) \rightarrow X$, since χ is dominant, we have $c_1(L_\chi^*) \cdot [C'] \geq 0$. So we may assume that C' is not in a fibre. The projection of C' to X is a finite cover $\pi_P : C' \rightarrow C$ to its image C . We may choose a smooth projective curve C'' and a morphism $h : C'' \rightarrow C$ such that $\tilde{C} = C'' \times_C C'$ is a split unramified covering. Then every sheet C_j of \tilde{C} is the image of a section σ_j of $\mathfrak{R}_P(h^*\mathfrak{E})$. Since $h^*\mathfrak{E}$ is semistable by Lemma 3.8, we have $\deg \sigma_j^*(L_\chi^*) \geq 0$ by Lemma 3.9. This implies (i).

Finally, we prove that (iii) implies (ii). $\mathrm{Ad}(\mathfrak{E})$ is semistable by Lemma 4.3; thus, since $c_2(\mathrm{Ad}(E)) = 0$, by Theorem A.5 the Higgs vector bundle $\mathrm{Ad}(f^*(\mathfrak{E}))$ is semistable, and then $f^*(\mathfrak{E})$ is semistable by Lemma 4.3. \square

Remark 4.8. For non-Higgs principal bundles, one actually proves that condition (iii) in Theorem 4.7 is equivalent to conditions (i) and (ii) [4].

Corollary 4.9. Assume that $\mathfrak{E} = (E, \phi)$ is a principal Higgs G -bundle, $\lambda : G \rightarrow G'$ is a surjective group homomorphism, $\mathfrak{E}' = (E', \phi')$ is a principal Higgs G' -bundle, and $f : E \rightarrow E'$ is a λ -equivariant morphism of principal Higgs bundles. If \mathfrak{E} satisfies condition (i) or (ii) of Theorem 4.7, so does \mathfrak{E}' .

Proof. If P' is a parabolic subgroup of G' , then $P' = \lambda(P)$ for a parabolic P in G . If $\chi' : P' \rightarrow \mathbb{C}^*$ is a dominant character of P' , the composition $\chi = \chi' \circ \lambda$ is a dominant character of P . If $f : E/P \rightarrow E'/P'$ is the induced morphism, we know that $f(\mathfrak{R}_P(\mathfrak{E})) \subset \mathfrak{R}_{P'}(\mathfrak{E}')$, so that $f^*(L_{\chi'|_{\mathfrak{R}_{P'}(\mathfrak{E}')}}^*) \simeq L_{\chi|_{\mathfrak{R}_P(\mathfrak{E})}}^*$. Since $L_{\chi|_{\mathfrak{R}_P(\mathfrak{E})}}^*$ is nef, and $f : \mathfrak{R}_P(\mathfrak{E}) \rightarrow \mathfrak{R}_{P'}(\mathfrak{E}')$ is surjective, $L_{\chi'|_{\mathfrak{R}_{P'}(\mathfrak{E}')}}^*$ is nef as well [14]. \square

In [8] we introduced a notion of numerically flat Higgs vector bundle (see also Appendix A of this paper). A special class of semistable principal Higgs bundles provides examples of such bundles.

Theorem 4.10. Let \mathfrak{E} be a principal Higgs bundle $\mathfrak{E} = (E, \phi)$ on a polarised smooth complex projective variety X . If \mathfrak{E} is semistable and $c_2(\mathrm{Ad}(E)) = 0$ in $H^4(X, \mathbb{R})$, then the adjoint Higgs bundle $\mathrm{Ad}(\mathfrak{E})$ is H-nflat.

Proof. At first we prove this theorem when X is a curve. In this case actually we can prove that \mathfrak{E} is semistable if and only if $\mathrm{Ad}(\mathfrak{E})$ is H-nflat. In view of Lemma 4.3, this amounts to proving that $\mathrm{Ad}(\mathfrak{E})$ is semistable if and only if it is H-nflat. Since $c_1(\mathrm{Ad}(E)) = 0$ this holds true (Lemma A.7 and Proposition A.8, see also [8, Corollaries 3.4 and 3.6]).

Let us assume now that $\dim(X) > 1$. If condition (i) holds, then $\mathfrak{E}|_C$ is semistable for any embedded curve C (as usual, if C is not smooth one replaces it with its normalisation). Thus $\mathrm{Ad}(\mathfrak{E})|_C$ is semistable, hence H-nflat. But this implies that $\mathrm{Ad}(\mathfrak{E})$ is H-nflat as well. \square

Remark 4.11. For non-Higgs principal bundles, one is able to prove that the two conditions in the statement of Theorem 4.10 are equivalent [5]. This characterisation shows that the numerically flat principal G -bundles defined in [7] for semisimple structure groups G are no more than the class of principal bundles singled out by one of the conditions of Theorem 4.7; cf. [7, Theorem 2.5], and Propositions 5.10 and 5.12.

5. Numerically effective principal (Higgs) bundles

In this section we wish to give a definition of numerical effectiveness and numerical flatness for principal (Higgs) bundles on a complex projective manifold X , and prove its main properties.

We start with some group-theoretic considerations. Given a complex reductive algebraic group G , let $P \subset G$ be a parabolic subgroup, and $R_u(P)$ the unipotent radical of P . A subgroup L of G such that $L \simeq P/R_u(P)$, and P is a semidirect product $P = LR_u(P)$, is called a *Levi factor* of P . All Levi factors are conjugated by elements of $R_u(P)$, and are reductive algebraic groups, whose root system is in general reducible; hence a Levi factor L may be written as $L = L_1 \cdots L_m$ according to the decomposition of its root system [16, Section 27.5].

Now let $\rho: G \rightarrow \mathrm{GL}(V)$ be a faithful rational representation, let W be a subspace of V , and let P be a maximal parabolic subgroup of G which stabilises W . There is an induced action of P on V/W . A factor L_i of the Levi group of P is said to be a *standard quotient* of P if ρ maps it injectively into $\mathrm{GL}(V/W)$ for some choice of ρ and W .

We may now define a notion of universal quotient bundle of a principal Higgs bundle. Let $\mathfrak{E} = (E, \phi)$ be a principal Higgs G -bundle on a projective manifold X . For any closed subgroup $K \subset G$, denote by E_K the principal K -bundle $E \rightarrow E/K$. (Recall that the restriction of E_K to the scheme of Higgs reductions $\mathfrak{R}_K \subset E/K$ carries an induced Higgs field, cf. Remark 3.6, thus giving rise to a principal Higgs K -bundle \mathfrak{E}_K .) If $P \subset G$ is a parabolic subgroup, and $\psi: P \rightarrow Q$ the projection onto a standard quotient, we call E_Q the principal Q -bundle obtained by extending the structure group of E_P to Q .

Definition 5.1. A universal Higgs quotient \mathfrak{E}_Q of \mathfrak{E} is the restriction of E_Q to the scheme of Higgs reductions $\mathfrak{R}_P(\mathfrak{E}) \subset E/P$, equipped with the Higgs field induced by the Higgs field of \mathfrak{E}_P . Here P is a maximal parabolic subgroup of G , and Q is a standard quotient of P .

Remark 5.2. The motivation for this definition is as follows. If G is the general linear group $\mathrm{GL}(V)$, where V is a complex finite-dimensional vector space, a maximal parabolic subgroup P in G stabilises a subspace $W \subset V$. Then a standard quotient of P is isomorphic to the group $\mathrm{GL}(V/W)$. If U is a vector bundle on a variety X , and E is the bundle of linear frames of U , the principal Q -bundle obtained by extending the structure group of E_P to Q is the bundle of linear frames of the universal rank k quotient bundle on the Grassmannian bundle E/P , where $k = \dim(V/W)$.

Remark 5.3. Note that this construction is functorial: if $f: Y \rightarrow X$ is a morphism of projective manifolds, then $(f^*\mathfrak{E})_Q \simeq \tilde{f}^*\mathfrak{E}_Q$, where $\tilde{f}: \mathfrak{R}_P(f^*\mathfrak{E}) \rightarrow \mathfrak{R}_P(\mathfrak{E})$ is the morphism induced by f .

We give now our definition of numerical effectiveness. This will be a recursive definition, with recursion on the semisimple rank of the structure group, and we start by defining numerical effectiveness for what will be the “terminal” case, i.e., principal Higgs T -bundles, where T is an algebraic torus.

Definition 5.4. Let $\mathfrak{E} = (E, \phi)$ be a principal Higgs T -bundle, with $\dim T = r$.

- (i) \mathfrak{E} is *Higgs-numerically effective* (H-nef for short) if there exists an isomorphism $\lambda : T \rightarrow (\mathbb{C}^*)^r$ such that the vector bundle associated to E via λ is nef.
- (ii) \mathfrak{E} is *Higgs-numerically flat* (H-nflat for short) if there exists an isomorphism $\lambda : T \rightarrow (\mathbb{C}^*)^r$ such that the vector bundle V_λ associated to E via λ is numerically flat, i.e., both V_λ and V_λ^* are numerically effective.

Higgs-numerical flatness can be equivalently defined by asking that the vector bundle associated to E via *any* isomorphism $T \rightarrow (\mathbb{C}^*)^r$ is numerically flat. Note that these definitions are independent of the Higgs field.

Let $D(G)$ be the derived subgroup of G . The quotient $R' = G/D(G)$ is isomorphic to the quotient of the radical R of G by a finite subgroup, and is therefore isomorphic to R . Let $\text{rad} : G \rightarrow R$ be the projection.

Definition 5.5. The radical of a principal Higgs G -bundle \mathfrak{E} is the principal Higgs R -bundle $R(\mathfrak{E}) = \mathfrak{E} \times_{\text{rad}} R \simeq \mathfrak{E}/D(G)$.

If \mathfrak{E} is the bundle of linear frames of a Higgs vector bundle $\mathfrak{V} = (V, \phi)$, then $R(\mathfrak{E})$ is the bundle of linear frames of the determinant line bundle $\det(V)$ equipped with the induced Higgs field $\det(\phi)$.

Proposition 5.6. *The radical $R(\mathfrak{E})$ of a principal Higgs G -bundle \mathfrak{E} is trivial (as a principal Higgs bundle, see Definition 3.2) if and only if \mathfrak{E} admits a Higgs reduction of its structure group to its derived subgroup $D(G)$.*

Proof. If $R(\mathfrak{E})$ is trivial, the principal R -bundle $E/D(G)$ is trivial, so that the structure group of E may be reduced to $D(G)$; let us denote by E' the reduced bundle. Since the Higgs field of $R(\mathfrak{E})$ is zero, the Higgs field ϕ of \mathfrak{E} is actually a section of $\text{Ad}(E') \otimes \Omega_X^1$, so that $\mathfrak{E}' = (E', \phi)$ is a Higgs reduction of the structure group of \mathfrak{E} to $D(G)$.

Conversely, if such a reduction exists, $R(E) = E/D(G)$ is trivial as it has a global section, and since ϕ lies in $\Gamma(\text{Ad}(E') \otimes \Omega_X^1)$, the Higgs field of $R(\mathfrak{E})$ vanishes. \square

Definition 5.7. A principal Higgs G -bundle \mathfrak{E} on X is H-nef if

- (i) $R(\mathfrak{E})$ is H-nef according to Definition 5.4;
- (ii) if $\text{rk}_{ss}(G) > 0$, for every maximal parabolic subgroup P and every standard quotient Q of P , the universal Higgs quotient \mathfrak{E}_Q is H-nef.

Moreover, \mathfrak{E} is said to be H-nflat if it is H-nef and $R(\mathfrak{E})$ is H-nflat.

Since the semisimple rank of the structure group Q of \mathfrak{E}_Q is strictly smaller than the semisimple rank of G , this recursive definition makes sense. As far as we know, this definition is new even in the case of (non-Higgs) principal bundles.

Remark 5.8. (i) If \mathfrak{E} is the bundle of linear frames of a Higgs vector bundle \mathfrak{V} , then, in view of Remark 5.2, it is H-nef (H-nflat) if and only if \mathfrak{V} is H-nef (H-nflat) in the sense of Definition A.2. As a further particular case, when the Higgs field is zero, so that we are dealing with an ordinary principal $GL(n, \mathbb{C})$ -bundle, the latter is nef in this sense if and only if the associated vector bundle is nef in the usual way.

(ii) Definition 5.7 implies that a principal Higgs G -bundle is H-nef if and only if $f^*\mathfrak{E}$ is H-nef for all morphisms $f: C \rightarrow X$ where C is a smooth algebraic curve.

We prove some basic properties of H-nef principal Higgs bundles.

Proposition 5.9.

- (i) *The pullback of an H-nef principal Higgs bundle is H-nef.*
- (ii) *A trivial Higgs G -bundle is H-nflat.*

Proof. Point (i) follows immediately from Remark 5.3, or from Remark 5.8(ii). The proof of point (ii) needs the following preliminary result.

Let G be a reductive linear algebraic group, $P \subset G$ a maximal parabolic subgroup, and let E_G be the principal G -bundle over G/P obtained by extending the structure group of the principal P -bundle $G \rightarrow G/P$ to G via the inclusion $P \rightarrow G$. One easily checks that E_G is trivial. Let \mathfrak{E}_G be E_G equipped with the trivial Higgs field. Then \mathfrak{E}_G is H-nef.

We prove this by induction the semisimple rank of G . If $\text{rk}_{ss}(G) = 0$, then \mathfrak{E}_G is the bundle of linear frames of a trivial Higgs vector bundle on G/P , so that it is H-nef (cf. Remark 5.8(i)).

If $\text{rk}_{ss}(G) > 0$, we first prove that $R(\mathfrak{E}_G)$ is H-nef. Let $\chi: R(G) \rightarrow \mathbb{C}^*$ be a character of the radical of G . The associated Higgs \mathbb{C}^* -bundle is trivial, hence H-nef by Remark 5.8(i), and then $R(\mathfrak{E}_G)$ is H-nef.

The inductive step is used to prove that the universal Higgs quotients of \mathfrak{E}_G are H-nef. Let $P' \subset G$ be a maximal parabolic subgroup of G , and let $\psi': P' \rightarrow Q'$ be the projection onto a standard quotient. The associated universal principal Higgs quotient is the pullback of the universal quotient $G \times_{\psi'} Q'$ via the projection $G/P \times G/P' \rightarrow G/P'$ (with the zero Higgs field). Now, $G \times_{\psi'} Q'$ is H-nef by the inductive hypothesis, and its pullback is H-nef due to point (i) of this proposition. So we have proved the inductive step.

Now we go back to the proof of point (ii). If $\mathfrak{E} = X \times G \rightarrow X$ with trivial Higgs field, then $R(\mathfrak{E}) \simeq X \times R(G)$ is the bundle of linear frames of a trivial Higgs vector bundle on X , so that it is H-nflat. Moreover, let $P \subset G$ be a parabolic subgroup, and Q its standard quotient. Then the associated universal quotient of \mathfrak{E}_G is the pullback of the universal quotient of the bundle $G \rightarrow G/P$ via the projection $X \times G/P \rightarrow G/P$, hence is H-nef due to point (i) and to the result we have previously proved. Thus \mathfrak{E} is H-nef, and since $R(\mathfrak{E})$ is H-nflat, \mathfrak{E} is also H-nflat. \square

Numerically flat principal Higgs bundles turn out to be semistable.

Proposition 5.10. *An H-nflat principal Higgs G -bundle \mathfrak{E} is semistable.*

Proof. Let $P \subset G$ be a maximal parabolic subgroup, and χ a nontrivial dominant character of P . Let Q be a standard quotient of P , and $\psi: P \rightarrow Q$ the projection. Given a character $\chi_Q: Q \rightarrow \mathbb{C}^*$ we may define a character χ' of P by letting $\chi' = \chi_Q \circ \psi$.

Since the universal quotient \mathfrak{E}_Q is an H-nef principal Higgs Q -bundle, the radical bundles $R(\mathfrak{E}_Q)$ are H-nef as well, and we may choose the character $\chi_Q: Q \rightarrow \mathbb{C}^*$ so that the restriction of the dual of the line bundle $L_Q = E_Q \times_{\chi_Q} \mathbb{C}$ to $\mathfrak{R}_P(\mathfrak{E}) \subset E/P$ is nef (cf. Definition 5.4: χ_Q may be taken as the composition of the isomorphism λ with the determinant morphism $(\mathbb{C}^*)^r$). Let L' be the line bundle on E/P associated to E_P by the character χ' . One defines a morphism

$$\begin{aligned} L' &\rightarrow L_Q, \\ (g, z) &\mapsto ((g, e), z) \end{aligned}$$

which turns out to be surjective, hence it is an isomorphism. Since $\text{Pic}(G/P) \simeq \mathbb{Z}$, we have $m_1\chi = m_2\chi' + \chi_0$, for some integers m_1, m_2 and a character χ_0 of the centre of G . The line bundle L_χ^* is nef when restricted to the fibres of $E/P \rightarrow X$ (which are copies of G/P), while L_Q^* is nef after restricting to the intersections of these fibres with the scheme of Higgs reductions $\mathfrak{R}_P(\mathfrak{E})$, and the restriction of the line bundle associated to χ_0 is numerically flat. Hence we may assume that m_1 and m_2 are both positive. Therefore $L_{\chi|\mathfrak{R}_P(\mathfrak{E})}^*$ is nef. This by Theorem 4.7 implies the claim. \square

Remark 5.11. For non-Higgs principal bundles, one can prove that a principal G -bundle E is numerically flat if and only if it is semistable and $c_2(\text{Ad}(E)) = 0$.

Proposition 5.12. *If a principal Higgs G -bundle \mathfrak{E} is semistable, satisfies $c_2(\text{Ad}(E)) = 0$, and its radical $R(\mathfrak{E})$ is H-nflat, then it is H-nflat.*

Proof. Since by hypothesis $R(\mathfrak{E})$ is H-nflat, we only need to show that all universal quotient principal Higgs bundles \mathfrak{E}_Q are H-nef. In particular, in virtue of our recursive definition, we need to show that all radicals $R(\mathfrak{E}_Q)$ are H-nef, and that a number of other radicals are H-nef as well (cf. Proposition A.4). Let us just check why the radicals $R(\mathfrak{E}_Q)$ are H-nef. Now, it turns out that every character of Q composed with the projection $\psi_s: P \rightarrow Q$ is a (possibly rational) multiple of a dominant character χ of P . Since \mathfrak{E} is semistable, and $c_2(\text{Ad}(E)) = 0$, by Theorem 4.7, the line bundle L_χ^* is nef. This implies the existence of an isomorphism $R(Q) \xrightarrow{\sim} (\mathbb{C}^*)^r$ such that the vector bundle associated by it to $R(\mathfrak{E}_Q)$ is nef. This means that $R(\mathfrak{E}_Q)$ is H-nef. \square

6. Numerically flat principal bundles and flat reductions

In [12] numerically flat vector bundles were characterised as vector bundles admitting filtrations whose quotients are locally free and stable, and admit flat unitary connections. In this section we prove a similar result for principal bundles, with a partial generalisation to principal Higgs bundles.

We start by reviewing some facts about connections on principal bundles, covering also the case when a Higgs field is present. Let K be a maximal compact subgroup of a connected reductive complex algebraic group G . Note that the Lie algebra \mathfrak{g} of G admits an involution ι , called the *Cartan involution*, whose $+1$ eigenspace is the Lie algebra \mathfrak{k} of K . If $\mathfrak{E} = (E, \phi)$ is a princi-

pal Higgs G -bundle, we may extend ι to an involution on the sections of the bundle $\mathrm{Ad}(E) \otimes \mathcal{A}^1$ (where \mathcal{A}^1 is the bundle of complex-valued smooth differential 1-forms) by letting

$$\iota(s \otimes \omega) = -\iota(s) \otimes \bar{\omega}.$$

Given a reduction σ of the structure group of E to K , there is a unique connection ∇_σ on E which is compatible with the complex structure of E and with the reduction [21]. By analogy with the vector bundle case, we call it the *Chern connection* associated with the reduction σ . The Higgs field may be used to introduce another connection

$$\nabla_{\sigma, \phi} = \nabla_\sigma + \phi + \iota(\phi)$$

which we call the *Hitchin–Simpson connection* of the triple $(\mathfrak{E}, \sigma) = (E, \phi, \sigma)$.

Definition 6.1. A principal Higgs G -bundle \mathfrak{E} is said to be *Hermitian flat* if it admits a reduction of its structure group to K such that the corresponding Hitchin–Simpson connection is flat.

To state our results we need the notion of *polystable* principal Higgs bundle. Let us recall that the notion of *slope* of a reduction was introduced in Section 3, cf. Lemma 3.9.

Definition 6.2. A reduction σ of the structure group of G of a principal Higgs G -bundle \mathfrak{E} to a parabolic subgroup $P \subset G$ is said to be *admissible* if $\mu_\sigma(\chi) = 0$ for every character χ of P which vanishes on the centre of G .

Definition 6.3. A principal Higgs G -bundle \mathfrak{E} is said to be *polystable* if there is a parabolic subgroup P of G and a Higgs reduction σ of the structure group of E to a Levi subgroup L of P such that

- (i) the reduced principal Higgs L -bundle \mathfrak{E}_σ is stable;
- (ii) the principal Higgs P -bundle obtained by extending the structure group of \mathfrak{E}_σ to P is an admissible reduction of the structure group of \mathfrak{E} to P (cf. Definition 6.2).

Also in this case one has a *Hitchin–Kobayashi correspondence* [21]. Choose a Kähler form ω on X representing the polarisation H we are using. We say that a reduction σ of the structure group G of a principal Higgs G -bundle \mathfrak{E} to a maximal compact subgroup K is *Hermitian–Yang–Mills* if there is an element τ in the centre \mathfrak{z} of the Lie algebra \mathfrak{g} of G such that

$$\mathcal{K}_{\sigma, \phi} = \tau$$

where $\mathcal{K}_{\sigma, \phi}$ is the *mean curvature* of the Hitchin–Simpson connection (computed with the Kähler form ω).

Theorem 6.4. (See [2].) *A principal Higgs G -bundle \mathfrak{E} is polystable if and only if it admits a Hermitian–Yang–Mills reduction to a maximal compact subgroup $K \subset G$.*

This notion of polystability extends the one holding for Higgs vector bundles, i.e., a Higgs vector bundle is polystable if it is a direct sum of stable Higgs vector bundles having the same

slope. A result similar to Lemma 4.3(i) may be proved. The proof of this result is implicitly contained in [2].

Proposition 6.5. *A principal Higgs bundle is polystable if and only if its adjoint bundle is polystable.*

We state now our second main result in the case of (non-Higgs) principal bundles.

Theorem 6.6. *A principal G -bundle E is nflat if and only if there is a parabolic subgroup P of G and a reduction σ of the structure group of E to P such that the principal $L(P)$ -bundle obtained by extending the structure group of the reduced bundle E_P to the Levi factor $L(P)$ is Hermitian flat and polystable.*

Proof. The “if” part is quite easily proved. Since E admits a flat connection, we have $c_2(\mathrm{Ad}(E)) = 0$. Moreover, E is polystable, hence semistable. The radical $R(E)$ carries an induced flat connection. Hence Proposition 5.12 implies that E is nflat.

Let us now prove the “only if” part. In view of Remark 4.11, we know that $\mathrm{Ad}(E)$ is nflat. As showed in [12], this implies that it has a filtration

$$0 \subset S_0 \subset \cdots \subset S_m = \mathrm{Ad}(E) \quad (1)$$

such that every quotient S_{i+1}/S_i is locally free, flat and stable. The analysis made in [2] (see also [5]) may be carried over to the present situation: one shows that the filtration (1) has an odd number of terms, and the middle term (say, S_ℓ) is isomorphic to the adjoint bundle $\mathrm{Ad}(F)$ of a reduction F of E whose structure group is a parabolic subgroup P of G .

Let E_L be the principal $L(P)$ -bundle obtained by extending the structure group of F to $L(P)$. It turns out that $\mathrm{Ad}(E_L)$ is isomorphic to the quotient $S_\ell/S_{\ell-1}$. Since the successive quotients of the filtration (1) are stable and flat, the bundle $\mathrm{Ad}(E_L)$ is stable, and moreover, all its Chern classes vanish [12]. The polystability of $\mathrm{Ad}(E_L)$ implies the polystability of E_L (see Proposition 6.5). By Theorem 6.4, E_L admits a reduction to the maximal compact subgroup of $L(P)$ such that the corresponding Chern connection satisfies the Hermitian–Yang–Mills condition.

Now, the homomorphism

$$L \rightarrow \mathrm{Aut}(\mathfrak{l}) \times R(L)$$

given by the adjoint representation of $L = L(P)$, and the projection onto the radical $R(L)$, gives an injective Lie algebra homomorphism

$$\mathfrak{l} \rightarrow \mathrm{End}(\mathfrak{l}) \oplus \mathfrak{r}(L). \quad (2)$$

Here \mathfrak{l} and $\mathfrak{r}(L)$ are the Lie algebras of L and $R(L)$, respectively. Thus we have a vector bundle $V = \mathrm{Ad}(E_L) \oplus W$ which is associated to E_L , and by Lemma 4.3 is semistable. Then $\deg(W) = \deg(\mathrm{Ad}(E_L)) = 0$. Moreover, V satisfies $\Delta(V) = 0$ because $\Delta(V)$ is a multiple of $c_2(\mathrm{Ad}(E_L))$. On the other hand, by the same reason we have $\Delta(W) = 0$. This implies $c_1(W)^2 = 0$.

The Hermitian–Yang–Mills connection on E_L induces Hermitian–Yang–Mills connections on $\mathrm{Ad}(E_L)$ and W . Lemma IV.4.12 of [17] (with the conditions $\deg(W) = 0$, $c_1(W)^2 = 0$) implies that the connection on \mathfrak{W} is flat, and the same is true for $\mathrm{Ad}(E_L)$. Since the morphism (2) is injective, the Hermitian–Yang–Mills connection on E_L is flat as well. \square

Corollary 6.7. *If E is nflat, the cohomology ring of E with coefficients in \mathbb{R} is trivial.*

Proof. This follows from the fact that the principal Higgs G -bundle obtained by extending the structure group of E_L to G is isomorphic to E as a topological bundle. Keeping up with the notation of Theorem 6.6, let E be a principal G -bundle, F a reduced bundle with structure group a parabolic subgroup $P \subset G$, and E_L the principal L -bundle obtained by extending the structure group of F to L (here L is the Levi group corresponding to P). Moreover, let E' be the G -bundle obtained by extending the structure group of E_L to G ; so, E' is the “graded object” corresponding to the reduction of G to P . Let $\rho: G \rightarrow \text{Aut}(W)$ be a faithful representation of G , and let $V = E \times_{\rho} W$ be the associated vector bundle. There exists a flag $0 = W_0 \subset \dots \subset W_{\ell} = W$ which is preserved by $\rho(P)$, such that the unipotent radical of P acts trivially on the quotients W_i/W_{i-1} . Thus $\rho(P)$ is contained in a parabolic subgroup P' of $\text{Aut}(W)$, and $\rho(L)$ is contained in a Levi subgroup L' of P' . The graded module V' of the filtration of V corresponding to P' is isomorphic to the associated bundle $E' \times_{\rho} W$, and on the other hand it is topologically isomorphic to V . This implies that E and E' are topologically isomorphic. \square

Remark 6.8. By Remarks 4.8 and 4.11, the “if” part of Theorem 6.6 holds true also for principal Higgs bundles.

7. Some Tannakian considerations

In this section we place Theorem 4.7 into the framework of Tannakian categories. We recall (see e.g. [11]) that a neutral Tannakian category \mathbf{T} over a field \mathbb{k} is a rigid abelian (associative and commutative) \mathbb{k} -linear tensor category such that

- (i) for every unit object 1 in \mathbf{T} , the endomorphism space $\text{End}(1)$ is isomorphic to \mathbb{k} ;
- (ii) there is an exact faithful functor $\omega: \mathbf{T} \rightarrow \mathbf{Vect}_{\mathbb{k}}$, called a *fibre functor*.

Here $\mathbf{Vect}_{\mathbb{k}}$ is the category of vector spaces over \mathbb{k} . The standard example of a neutral Tannakian category is the category $\mathbf{Rep}(G)_{\mathbb{k}}$ of \mathbb{k} -linear representations of an affine group scheme G . Indeed, any neutral Tannakian category can be represented as $\mathbf{Rep}(G)_{\mathbb{k}}$ where G is the automorphism group of the fibre functor ω . Let \mathcal{E} be a principal Higgs G -bundle on a (say) complex projective manifold X . For any finite-dimensional linear representation $\rho: G \rightarrow \text{Aut}(W)$ let $\mathcal{W} = \mathcal{E} \times_{\rho} W$ be the associated Higgs vector bundle. This correspondence defines a G -torsor on the category \mathbf{Higgs}_X of Higgs vector bundles on X , i.e., a faithful and exact functor $\mathcal{E}: \mathbf{Rep}(G)_{\mathbb{k}} \rightarrow \mathbf{Higgs}_X$ [22]. In general, this is not always compatible with semistability, i.e., $\mathcal{E}(\rho, W)$ is not always semistable even when \mathcal{E} is. In order to have that, we need to impose some conditions. For instance, we may assume that every representation $\rho: G \rightarrow \text{Aut}(W)$ maps the connected component of the centre of G containing the identity to the centre of $\text{Aut}(W)$ (this happens, e.g., when G is semisimple). When this is true, we say that G is *central*.

Let $\mathbf{Higgs}_X^{\Delta}$ be the full subcategory of \mathbf{Higgs}_X whose objects \mathcal{W} are Higgs vector bundles such that $f^*\mathcal{W}$ is semistable for every morphism $f: C \rightarrow X$, where C is any smooth projective curve (in particular, such Higgs bundles are semistable). Since the tensor product of semistable Higgs bundles is semistable [22], $\mathbf{Higgs}_X^{\Delta}$ is a tensor category. However, it is not additive but only preadditive (i.e., every homomorphism set $\text{Hom}(\mathcal{V}, \mathcal{W})$ is an abelian group, and composition of morphisms is bilinear over the integers). Let $\mathbf{Higgs}_X^{\Delta,+}$ be its additive completion (see e.g. [13]). We may now prove the following characterisation.

Proposition 7.1. *Assume that G is central. There is one-to-one correspondence between principal Higgs G -bundles \mathfrak{E} satisfying condition (i) or (ii) of Theorem 4.7 and G -torsors on the category \mathbf{Higgs}_X taking values in $\mathbf{Higgs}_X^{\Delta,+}$.*

Proof. Given a principal Higgs G -bundle \mathfrak{E} and a representation $\rho : G \rightarrow \mathrm{Aut}(W)$ the associated Higgs vector bundle \mathfrak{W} is semistable by Lemma 4.3 (since G is central), and this is true after pullback to any curve.

Conversely, given a G -torsor on $\mathbf{Higgs}_X^{\Delta,+}$, one builds a principal Higgs G -bundle \mathfrak{E} as in [22, Ch. 6]. We prove that \mathfrak{E} is semistable. If \mathfrak{W} is an associated Higgs vector bundle via a faithful representation, $\mathrm{Ad}(\mathfrak{E})$ is a Higgs subbundle of $\mathrm{End}(\mathfrak{W})$. If \mathfrak{W} is semistable, since $c_1(\mathrm{Ad}(\mathfrak{E})) = c_1(\mathrm{End}(\mathfrak{W})) = 0$ the bundle $\mathrm{Ad}(\mathfrak{E})$ is semistable, so that \mathfrak{E} is semistable as well. This is true after pullback to any curve, so that \mathfrak{E} is semistable after pullback to any curve. \square

Remark 7.2. In the case of principal (non-Higgs) G -bundles, let us denote by $\mathbf{Vect}_X^{\Delta,+}$ the additive completion of the category of vector bundles that are semistable after pullback to any curve. Since in this case all three conditions in Theorem 4.7 are equivalent, $\mathbf{Vect}_X^{\Delta,+}$ is equivalent to the additive completion of the category of semistable vector bundles with vanishing discriminant.

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Appendix A. Semistable and numerically effective Higgs vector bundles

Since our treatment of principal Higgs bundles relies quite heavily on previous work on Higgs vector bundles, we provide here a short resume of the main results in that theory. The main references are [8–10], even though the treatment we give here includes some modifications. We shall give here only a sketch of the main proofs, referring to [8–10] for a more detailed and complete treatment.

A.1. Ample and numerically effective Higgs bundles

All varieties are projective varieties over the complex field. Let V be a vector bundle of rank r on X , and let s be a positive integer less than r . We shall denote by $\mathrm{Gr}_s(V)$ the Grassmann bundle of s -planes in V , with projection $p_s : \mathrm{Gr}_s(V) \rightarrow X$. There is a universal exact sequence

$$0 \rightarrow S_{r-s,V} \xrightarrow{\psi} p_s^*(V) \xrightarrow{\eta} Q_{s,V} \rightarrow 0 \quad (3)$$

of vector bundles on $\mathrm{Gr}_s(V)$, with $S_{r-s,V}$ the universal rank $r-s$ subbundle and $Q_{s,V}$ the universal rank s quotient bundle.

Definition A.1. A Higgs sheaf \mathfrak{V} on X is a coherent sheaf V on X endowed with a morphism $\phi: V \rightarrow V \otimes \Omega_X$ of \mathcal{O}_X -modules such that $\phi \wedge \phi = 0$, where Ω_X is the cotangent sheaf to X . A Higgs subsheaf W of a Higgs sheaf $\mathfrak{V} = (V, \phi)$ is a subsheaf of V such that $\phi(W) \subset W \otimes \Omega_X$. A Higgs bundle is a Higgs sheaf \mathfrak{V} such that V is a locally free \mathcal{O}_X -module. A Higgs sheaf $\mathfrak{V} = (V, \phi)$ is semistable (resp. stable) if V is torsion-free, and $\mu(W) \leq \mu(V)$ (resp. $\mu(W) < \mu(V)$) for every proper nontrivial Higgs subsheaf W of \mathfrak{V} .

Given a Higgs bundle \mathfrak{V} , we may construct closed subschemes $\mathfrak{Gr}_s(\mathfrak{V}) \subset \text{Gr}_s(V)$ parameterising rank s locally free Higgs quotients, i.e., locally free quotients of V whose corresponding kernels are ϕ -invariant. We define $\mathfrak{Gr}_s(\mathfrak{V})$ (the Grassmannian of locally free rank s Higgs quotients of \mathfrak{V}) as the closed subscheme of $\text{Gr}_s(V)$ where the composed morphism

$$(\eta \otimes 1) \circ p_s^*(\phi) \circ \psi: S_{r-s, V} \rightarrow Q_{s, V} \otimes p_s^* \Omega_X \quad (4)$$

vanishes. We denote by ρ_s the projections $\mathfrak{Gr}_s(\mathfrak{V}) \rightarrow X$. The restriction of (3) to the scheme $\mathfrak{Gr}_s(\mathfrak{V})$ provides the universal exact sequence $0 \rightarrow S_{r-s, \mathfrak{V}} \xrightarrow{\psi} \rho_s^*(V) \xrightarrow{\eta} Q_{s, \mathfrak{V}} \rightarrow 0$, and $Q_{s, \mathfrak{V}}$ is a rank s universal Higgs quotient vector bundle, i.e., for every morphism $f: Y \rightarrow X$ and every rank s Higgs quotient W of f^*V there is a morphism $\psi_W: Y \rightarrow \mathfrak{Gr}_s(\mathfrak{V})$ such that $f = \rho_s \circ \psi_W$ and $W \simeq \psi_W^*(Q_{s, \mathfrak{V}})$.

Definition A.2. A Higgs bundle \mathfrak{V} of rank one is said to be Higgs-numerically effective (H-nef) if it is numerically effective in the usual sense. If $\text{rk } \mathfrak{V} \geq 2$ we require that:

- (i) all bundles $Q_{s, \mathfrak{V}}$ are Higgs-nef;
- (ii) the line bundle $\det(V)$ is nef.

If both \mathfrak{V} and \mathfrak{V}^* are Higgs-numerically effective, \mathfrak{V} is said to be Higgs-numerically flat (H-nflat).

Note that if $\mathfrak{V} = (V, \phi)$, with V nef in the usual sense, then \mathfrak{V} is H-nef. Moreover, if $\phi = 0$, the Higgs bundle $\mathfrak{V} = (V, 0)$ is H-nef if and only if V is nef in the usual sense.

Proposition A.3. Let X be a smooth projective variety.

- (i) If $f: Y \rightarrow X$ is a finite surjective morphism of smooth projective varieties, and \mathfrak{V} is a Higgs bundle on X , then \mathfrak{V} is H-ample (resp. H-nef) if and only if $f^*\mathfrak{V}$ is H-ample (resp. H-nef).
- (ii) Every quotient Higgs bundle of an H-nef Higgs bundle \mathfrak{V} on X is H-nef.

The recursive condition in the definition of H-nefness may be actually expressed in terms of a simpler set of nefness conditions. Let us denote by $Q(s_1, \dots, s_k)_{\mathfrak{V}}$ the universal Higgs bundle obtained by taking the successive universal Higgs quotients of \mathfrak{V} , first of rank s_k , then s_{k-1} , all the way to rank s_1 . The indexes s_1, \dots, s_k satisfy

$$1 \leq s_1 < \dots < s_k < r. \quad (5)$$

So for instance, $Q_{1,Q_s,\mathfrak{V}} = Q(1,s)\mathfrak{V}$. Moreover $Q(s_1, \dots, s_k)\mathfrak{V}$ is a rank s_1 Higgs bundle on $\mathfrak{Gr}_{s_1}(Q(s_2, \dots, s_k)\mathfrak{V})$. The H-nefness condition for \mathfrak{V} amounts to saying that the determinant bundles $\det(Q(s_1, \dots, s_k)\mathfrak{V})$ are nef for all strings s_1, \dots, s_k satisfying (5), and that the line bundles $Q(1, s_2, \dots, s_k)\mathfrak{V}$ are nef, for all strings of the type $(1, s_2, \dots, s_k)$.

Proposition A.4. *Let \mathfrak{V} be a Higgs bundle on X . The following conditions are equivalent.*

- (i) \mathfrak{V} is H-nef.
- (ii) *For every s satisfying $0 < s < r = \text{rk}(\mathfrak{V})$ the line bundle $Q_{1,Q_s,\mathfrak{V}}$ on $\mathfrak{Gr}_1(Q_s,\mathfrak{V})$ is nef, and for every string of integers s_1, \dots, s_k such that $1 \leq s_1 < \dots < s_k < r$, the line bundles $\det(Q(s_1, \dots, s_k)\mathfrak{V})$ are nef.*

Proof. One has a (surjective) morphism

$$\rho_{s_2,\dots,s_k} : \mathfrak{Gr}_1(Q(s_2, \dots, s_k)\mathfrak{V}) \rightarrow \mathfrak{Gr}_{s_k}(\mathfrak{V}),$$

and $Q(1, s_2, \dots, s_k)\mathfrak{V}$ turns out to be a rank one Higgs quotient of $\rho_{s_2,\dots,s_k}^* Q_{s_k,\mathfrak{V}}$. By universality, there is morphism

$$f_{s_2,\dots,s_k} : \mathfrak{Gr}_1(Q(s_2, \dots, s_k)\mathfrak{V}) \rightarrow \mathfrak{Gr}_1(Q_{s_k},\mathfrak{V})$$

such that

$$Q(1, \dots, s_k)\mathfrak{V} \simeq f_{s_2,\dots,s_k}^* Q(1, s_k)\mathfrak{V}. \quad (6)$$

If \mathfrak{V} is H-nef, the nefness of the determinant bundles $\det(Q(s_1, \dots, s_k)\mathfrak{V})$ and of the line bundle $Q(1, s_k)\mathfrak{V}$ holds by definition.

Conversely, if the conditions in (ii) hold, the line bundles $Q(1, \dots, s_k)\mathfrak{V}$ are nef as a consequence of (6), so that \mathfrak{V} is H-nef. \square

A.2. Generalising Miyaoka's semistability criterion

In [18] Miyaoka introduced a numerical class λ in the projectivisation $\mathbb{P}V$ which, when X is a curve, is nef if and only if V is semistable. In the case of a Higgs bundle \mathfrak{V} on a smooth projective variety X , we introduce the following generalisations of the class λ . These are numerical classes in the Higgs Grassmannians $\mathfrak{Gr}_s(\mathfrak{V})$:

$$\theta_{s,\mathfrak{V}} = [c_1(Q_{s,\mathfrak{V}})] - \frac{s}{r} \rho_s^*(c_1(V)) \in N^1(\mathfrak{Gr}_s(\mathfrak{V})),$$

where $\rho_s : \mathfrak{Gr}_s(\mathfrak{V}) \rightarrow X$ is the natural epimorphism.

Let $\Delta(V)$ be the characteristic class

$$\Delta(V) = c_2(V) - \frac{r-1}{2r} c_1(V)^2 = \frac{1}{2r} c_2(V \otimes V^*).$$

Theorem A.5. *Let \mathfrak{V} be a Higgs bundle on a smooth projective variety. Consider the following conditions.*

- (i) All classes $\theta_{s,\mathfrak{V}}$ are nef, for $0 < s < r$.
- (ii) For any smooth projective curve C in X , the restriction $\mathfrak{V}|_C$ is semistable.
- (iii) \mathfrak{V} is semistable and $\Delta(V) = 0$.

Then conditions (i) and (ii) are equivalent, and they are both implied by condition (iii).

Lemma A.6. A Higgs bundle \mathfrak{V} on a smooth projective curve C is semistable if and only if all classes $\theta_{s,\mathfrak{V}}$ are nef.

Proof. Assume \mathfrak{V} is semistable. If for some s the class $\theta_{s,\mathfrak{V}}$ is not nef there is an irreducible curve $C' \subset \mathfrak{Gr}_s(\mathfrak{V})$ which surjects onto C and is such that $C' \cdot \theta_{s,\mathfrak{V}} < 0$. As in the proof of Theorem 4.7, we may assume that $C' \rightarrow C$ is an isomorphism. Denote by Ω the restriction of $Q_{s,\mathfrak{V}}$ to C' , and let $\mathfrak{V}' = (p_s^* \mathfrak{V})|_{C'}$, where $p_s : \mathfrak{Gr}_s(\mathfrak{V}) \rightarrow C$ is the projection. \mathfrak{V}' is semistable, and we have

$$0 > [C'] \cdot \theta_{s,\mathfrak{V}} = [C'] \cdot \left(c_1(\Omega) - \frac{s}{r} p_s^* c_1(V) \right) = s(\mu(\Omega) - \mu(\mathfrak{V}'))$$

but this contradicts the semistability of \mathfrak{V}' .

If all classes $\theta_{s,\mathfrak{V}}$ are nef, let \mathfrak{V}' be a rank s Higgs quotient of \mathfrak{V} , and let $\sigma : C \rightarrow \mathfrak{Gr}_s(\mathfrak{V})$ be the corresponding section. Then

$$0 \leq \theta_{s,\mathfrak{V}} \cdot [\sigma(C)] = s(\mu(V') - \mu(V))$$

so that \mathfrak{V} is semistable. \square

This implies that conditions (i) and (ii) in Theorem A.5 are equivalent.

Lemma A.7. Let \mathfrak{V} be a Higgs bundle on a smooth projective variety X . If the restriction of \mathfrak{V} to any smooth curve C in X is semistable, and $c_1(V) = 0$, then \mathfrak{V} is H - η -flat.

Proof. We may assume that X is a curve. Let

$$\lambda_{s,\mathfrak{V}} = c_1(\mathcal{O}_{\mathbb{P}^{Q_{s,\mathfrak{V}}}}(1))|_{\mathfrak{Gr}_1(Q_{s,\mathfrak{V}})}.$$

We show that for every s , with $0 < s < r = \text{rk}(\mathfrak{V})$, the class $\lambda_{s,\mathfrak{V}}$ is nef. Let C be a curve in $\mathfrak{Gr}_1(Q_{s,\mathfrak{V}})$. Possibly after a base change, we may assume that C projects isomorphically onto a curve C' in $\mathfrak{Gr}_s(\mathfrak{V})$ and that this projects isomorphically onto X . We have the diagram

$$\begin{array}{ccc} C & \xrightarrow{j} & \mathfrak{Gr}_1(Q_{s,\mathfrak{V}}) \\ \sigma \uparrow & & \downarrow \\ C' & \xrightarrow{\quad} & \mathfrak{Gr}_s(\mathfrak{V}) \\ & \swarrow \sigma' & \downarrow \\ & & X. \end{array}$$

If we let $\mathcal{L} = (j \circ \sigma \circ \sigma')^* \mathcal{O}_{\mathbb{P}Q_{s,\mathfrak{Y}}}(1)$ then \mathcal{L} is a rank one Higgs quotient of \mathfrak{Y} , so that $\mu(\mathfrak{Y}) \leq \deg(\mathcal{L})$. But since $\deg(\mathcal{L}) - \mu(\mathfrak{Y}) = [C] \cdot \lambda_{s,\mathfrak{Y}}$, we have that $\lambda_{s,\mathfrak{Y}}$ is nef.

Now we prove that \mathfrak{Y} is H-nflat. In this actually enough to prove that \mathfrak{Y} is H-nef, and then apply the same reasoning to \mathfrak{Y}^* . Now, $\lambda_{s,\mathfrak{Y}}$ is the first Chern class of the line bundle $Q_{1,Q_{s,\mathfrak{Y}}}$. In view of Proposition A.4, it remains only to show that the determinant bundles $\det(Q(s_1, \dots, s_k)_{\mathfrak{Y}})$ are nef for all strings of integers s_1, \dots, s_k as in Proposition A.4. We note that $Q(s_1, \dots, s_k)_{\mathfrak{Y}}$ is a Higgs bundle on $\mathfrak{Gr}_{s_1}(Q(s_2, \dots, s_k)_{\mathfrak{Y}})$ and that there is a morphism $\rho: \mathfrak{Gr}_{s_1}(Q(s_2, \dots, s_k)_{\mathfrak{Y}}) \rightarrow X$ such that $Q(s_1, \dots, s_k)_{\mathfrak{Y}}$ is a Higgs quotient of $\rho^*\mathfrak{Y}$. Therefore, by universality, there is morphism $g_{s_1, s_2, \dots, s_k}: \mathfrak{Gr}_{s_1}(Q(s_2, \dots, s_k)_{\mathfrak{Y}}) \rightarrow \mathfrak{Gr}_{s_1}(\mathfrak{Y})$ such that $Q(s_1, \dots, s_k)_{\mathfrak{Y}} \simeq g_{s_1, s_2, \dots, s_k}^* Q_{s_1, \mathfrak{Y}}$. We have now

$$c_1(Q(s_1, \dots, s_k)_{\mathfrak{Y}}) = g_{s_1, s_2, \dots, s_k}^* (c_1(Q_{s_1, \mathfrak{Y}})) = g_{s_1, s_2, \dots, s_k}^* \theta_{s_1, \mathfrak{Y}}$$

since $c_1(V) = 0$, so that $\det(Q(s_1, \dots, s_k)_{\mathfrak{Y}})$ is nef since $\theta_{s_1, \mathfrak{Y}}$ is nef by Lemma A.6. \square

Proposition A.8. *An H-nflat Higgs bundle \mathfrak{Y} on a smooth projective variety X is semistable.*

Proof. As the restriction of an H-nflat Higgs bundle \mathfrak{Y} to a closed subvariety of X is H-nflat, we may assume that X is a curve. Since \mathfrak{Y} is in particular H-nef, all universal Higgs quotients $Q_{s,\mathfrak{Y}}$ are H-nef, and then the determinant bundles $\det Q_{s,\mathfrak{Y}}$ are nef. On the other hand, since $\det(V)$ is numerically flat, we have $c_1(V) = 0$. Therefore the classes $\theta_{s,\mathfrak{Y}}$ are nef. We conclude by Lemma A.6. \square

To conclude the proof of Theorem A.5 one needs to prove that condition (iii) implies condition (ii). This is proved in [10].

Remark A.9. For (non-Higgs) vector bundles, the three conditions in Theorem A.5 are equivalent [10].

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